

# LINEAR DECOMPOSITION ATTACK ON PUBLIC KEY EXCHANGE PROTOCOLS USING SEMIDIRECT PRODUCTS OF (SEMI)GROUPS

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ABSTRACT. We show that a linear decomposition attack based on the decomposition method introduced by the author in monography [1] and paper [2] works by finding the exchanging keys in the both two main protocols in [3] and [4].

## 1. INTRODUCTION

In this paper we present a new practical attack on two main protocols proposed in [3] and [4]. This kind of attack introduced by the author in [1] and [2] works when the platform groups are linear. We show that in this case, contrary to the common opinion (and some explicitly stated security assumptions), one does not need to solve the underlying algorithmic problems to break the scheme, i.e., there is another algorithm that recovers the private keys without solving the principal algorithmic problem on which the security assumptions are based. This changes completely our understanding of security of these scheme. The efficacy of the attack depends on the platform group, so it requires a specific analysis in each particular case. In general one can only state that the attack is in polynomial time in the size of the data, when the platform and related groups are given together with their linear representations. In many other cases we can effectively use known linear presentations of the groups under consideration. A theoretical base for the decomposition method is described in [5] where a series of examples is presented. The monography [1] solves uniformly protocols based on the conjugacy search problem (Ko, Lee et. al. [6], Wang, Cao et. al [7]), protocols based on the decomposition and factorization problems (Stickel [8], Alvares, Martinez et. al. [9], Shpilrain, Ushakov [10], Romanczuk, Ustimenko [11]), protocols based on actions by automorphisms (Mahalanobis [12], Rososhek [13], Markov, Mikhalev et. al. [14]), and a number of other protocols. See also [15] where the linear decomposition attack is applied to the two main protocols in [16].

In [4], D. Kahrobaei, H.T. Lam and V. Shpilrain described a public key exchange protocol based on an extension of a semigroup by automorphisms (more generally endomorphisms). They proposed a non-commutative semigroup of matrices over a Galois field as platform.

In this paper we present a polynomial time deterministic attack that breaks the two variants of the protocol presented in the papers [3] and [4].

All along the paper we denote by  $\mathbb{N}$  the set of all positive integers.

## 2. GENERAL KEY EXCHANGE PROTOCOL [3], [4].

In this section, we describe a not platform-specific key exchange protocol proposed in [3] and improved in [4]. We consider the more general version of this protocol presented in [4]. The corresponding version from [3] has been analyzed in [5]. Then we will give a cryptanalysis of this protocol under additional assumption of linearity of the chosen platform.

Let  $G$  be a (semi)group and  $g$  be a public element in  $G$ . Let  $\phi$  be an arbitrary public endomorphism of  $G$ . Let  $G_\phi = G \rtimes \text{sgp}(\phi)$  be the semidirect product of  $G$  and the semigroup  $\text{sgp}(\phi)$  generated by  $\phi$ . Recall that each element of  $G_\phi$  has a unique expression of the form  $(\phi^r, f)$  where  $r \in \mathbb{N} \cup \{0\}$  and  $f \in G$ . Two elements of this form are multiplied as follows:  $(\phi^r, f) \cdot (\phi^s, h) = (\phi^{r+s}, \phi^s(f)h)$ .

- Alice chooses a private  $m \in \mathbb{N}$ , while Bob chooses a private  $n \in \mathbb{N}$ .
- Alice computes  $(\phi, g)^m = (\phi^m, \phi^{m-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g)$  and sends only the second component  $a_m = \phi^{m-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g$  of this pair to Bob.
- Bob computes  $(\phi, g)^n = (\phi^n, \phi^{n-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g)$  and sends only the second component  $a_n = \phi^{n-1}(g) \cdots \phi^2(g) \cdot \phi(g) \cdot g$  of this pair to Alice.
- Alice computes  $(*, a_n)(\phi^m, a_m) = (*, \phi^m(a_n)a_m)$ . She does not actually "compute" the first component of the pair.
- Bob computes  $(*, a_m)(\phi^n, a_n) = (*, \phi^n(a_m)a_n)$ . He does not actually "compute" the first component of the pair.
- Since  $\phi^m(a_n)a_m = \phi^n(a_m)a_n = a_{m+n}$ , we should have  $K_{\text{Alice}} = K_{\text{Bob}} = a_{m+n}$ , the shared secret key.

This algorithm can be named the *noncommutative shift*.

Now we show how the shared secret key  $K = K_{\text{Alice}} = K_{\text{Bob}}$  can be computed in the case when  $G$  is a multiplicative subgroup of a finite dimensional algebra  $\mathbf{A}$  over a field  $\mathbb{F}$  and the endomorphism  $\phi$  is extended to an endomorphism of the underlying vector space  $V$  of  $\mathbf{A}$ . Furthermore, we assume that the basic field operations in  $\mathbb{F}$  are efficient, in particular they can be performed in polynomial time in the size of the elements, e.g.,  $\mathbb{F}$  is finite. In all the particular protocols considered in this paper the field  $\mathbb{F}$  satisfies all these conditions.

Using Gauss elimination we can effectively find a maximal linearly independent subset  $L$  of the set  $\{a_0, a_1, \dots, a_k, \dots\}$ , where  $a_0 = g$  and  $a_k = \phi^{k-1}(g) \cdots \phi(g) \cdot g$  for  $k \geq 1$ . Indeed, suppose that  $\{a_0, \dots, a_k\}$  is linearly independent set but  $a_{k+1}$  can be presented as a linear combination of the form

$$a_{k+1} = \sum_{i=0}^k \lambda_i a_i, \text{ for } \lambda_i \in \mathbb{F}.$$

Suppose by induction that  $a_{k+j}$  can be presented as above for every  $j \leq t-1$ . In particular

$$a_{k+t-1} = \sum_{i=0}^k \mu_i a_i, \text{ for } \mu_i \in \mathbb{F}.$$

Then

$$\begin{aligned} a_{k+t} &= \phi(a_{k+t-1}) \cdot g = \sum_{i=0}^k \mu_i \phi(a_i) \cdot g = \\ \sum_{i=0}^k \mu_i a_{i+1} &= \mu_k \lambda_0 a_0 + \sum_{i=0}^{k-1} (\mu_i + \mu_k \lambda_{i+1}) a_{i+1}. \end{aligned}$$

Thus  $L = \{a_0, \dots, a_k\}$ .

In particular, we can effectively compute

$$(1) \quad a_n = \sum_{i=0}^k \eta_i a_i, \text{ for } \eta_i \in \mathbb{F}.$$

Then

$$(2) \quad \begin{aligned} a_{m+n} &= \phi^m(a_n) \cdot a_m = \\ \sum_{i=0}^k \eta_i \phi^m(a_i) \cdot a_m &= \sum_{i=0}^k \eta_i \phi^i(a_m) \cdot a_i. \end{aligned}$$

Note that all data on the right hand side of (2) is known now. Thus we get the shared key  $K = a_{m+n}$ .

In the original version of this cryptosystem [3]  $G$  was proposed to be the semigroup of  $3 \times 3$  matrices over the group algebra  $\mathbb{F}_7[\mathbb{A}_5]$ , where  $\mathbb{A}_5$  is the alternating group on 5 elements. The authors of [3] used an extension of the semigroup  $G$  by an inner automorphism which is conjugation by a matrix  $H \in \text{GL}_3(\mathbb{F}_7[\mathbb{A}_5])$ . Therefore, in this case there is a polynomial time algorithm to find the shared key  $K$  from the public data.

### 3. KEY EXCHANGE PROTOCOL USING MATRICES OVER A GALOIS FIELD AND EXTENSIONS BY SPECIAL ENDOMORPHISMS [4].

In this section, we describe the key exchange protocol using matrices over a Galois field and extensions by special endomorphisms proposed in [4].

Let  $G$  be a multiplicative semigroup of the matrix algebra  $\mathbf{A} = \text{M}_2(\mathbb{F})$  of all  $2 \times 2$  matrices over the Galois field  $\mathbb{F} = \mathbb{F}_{2^{127}}$ . Let  $\varphi = \sigma_H$  be the automorphism of  $G$  which is a composition of a conjugation by a matrix

$H \in \text{GL}_2(\mathbb{F})$  with the endomorphism  $\psi$  that raises each entry of a given matrix to the power of 4. The composition is such that  $\psi$  is applied first, followed by conjugation. Note that both these maps naturally extend to automorphisms of  $\mathbf{A}$ .

This protocol can be attacked by the linear decomposition attack as it has been explained in Section 2.

In [4], the situation is considered where the automorphism  $\varphi$  is just conjugation by a public matrix  $H \in \text{GL}_2(\mathbb{F})$ . Let  $g = M \in G$ . By direct computation one get  $a_k = H^{-k}(HM)^k$  for every  $k \in \mathbb{N}$ .

This protocol is vulnerable to a linear algebra attack as follows. The attacker, Eve, is looking for matrices  $X$  and  $Y$  such that  $XH = HX$ ,  $Y(HM) = (HM)Y$ , and  $XY = H^{-m}(HM)^m$ . The first two matrix equations translate into a system of linear equations in the entries of  $X$  and  $Y$  over  $\mathbb{F}$ . After solving this system and finding invertible solution  $X$  and  $Y$ , Eve can recover the shared secret key  $K$  as follows:  $Xa_nY = H^{-n}(XY)(HM)^n = H^{-n}H^{-m}(HM)^m(HM)^n = H^{-(m+n)}(HM)^{m+n} = a_{m+n} = K$ . The above algorithm contains a couple of difficulties. Firstly, a solution  $X$  might be invertible. Secondly, all this computations should be done online during every session.

In contrast to the linear algebra attack, the linear decomposition attack is very simple. We describe even a more simple version of this attack working in this specific situation.

Consider the linear space  $W = \text{Sp}_{\mathbb{F}}(gp(H) \cdot sgp(HM))$  generated by all elements of the form  $H^k(HM)^l$  where  $k, l \in \mathbb{N} \cup \{0\}$ . One can find effectively a basis  $e_1, \dots, e_t$  of  $W$ . Obviously,  $t \leq 4$ . Moreover, since every matrix is a root of a characteristic polynomial of degree 2 one can choose basic elements in the form  $e_i = H^{k_i}(HM)^{l_i}$ ,  $k_i, l_i \in \{0, 1\}$ ,  $i = 1, \dots, t$ . Now we have public dates  $a_m$  and  $a_n$  where  $m, n \in \mathbb{N}$ . We can effectively compute

$$(3) \quad a_n = \sum_{i=1}^t \eta_i e_i = \sum_{i=1}^t \eta_i H^{-k_i}(HM)^{l_i}, \text{ for } \eta_i \in \mathbb{F}, i = 1, \dots, t.$$

Then

$$\begin{aligned} \sum_{i=1}^t \eta_i H^{-k_i} a_m (HM)^{l_i} &= \sum_{i=1}^t \eta_i H^{-k_i} (H^{-m}(HM)^m)(HM)^{l_i} = \\ &= H^{-m} \left( \sum_{i=1}^t \eta_i H^{-k_i} (HM)^{l_i} \right) (HM)^m = \\ (4) \quad &= H^{-m} H^{-n} (HM)^n (HM)^m = H^{-(m+n)} (HM)^{m+n} = a_{m+n}. \end{aligned}$$

Thus one has the shared key  $K = a_{m+n}$ . Note that the basis  $e_1, \dots, e_t$  is constructed one time offline. We don't need to look in any invertible solution.

In [4], the last protocol was changed to avoid the linear algebra attack. As before  $H, M \in G$ , where  $H$  is invertible and  $M$  is assumed to be not invertible. The automorphism  $\varphi$  is  $\sigma_H$ , the inner automorphism corresponding to  $H$ .

- Alice chooses a private  $m \in \mathbb{N}$ , while Bob chooses a private  $n \in \mathbb{N}$ . Alice also selects a private nonzero matrix  $R$  such that  $R \cdot (HM) = 0$  (the zero matrix), and Bob selects a private nonzero matrix  $S$  such that  $S \cdot (HM) = 0$ . Such matrices  $R, S$  exist because the matrix  $HM$  is not invertible.
- Alice computes  $(\varphi, M)^m = (\varphi^m, \varphi^{m-1}(M) \cdots \varphi^2(M) \cdot \varphi(M) \cdot M)$  where the second component of this pair is  $a_m = \varphi^{m-1}(M) \cdots \varphi^2(M) \cdot \varphi(M) \cdot M = H^{-m}(HM)^m$ , and sends  $a_m + R$  to Bob.
- Bob computes  $(\varphi, M)^n = (\varphi^n, \varphi^{n-1}(M) \cdots \varphi^2(M) \cdot \varphi(M) \cdot M)$ , where the second component is  $a_n = \varphi^{n-1}(M) \cdots \varphi^2(M) \cdot \varphi(M) \cdot M = H^{-n}(HM)^n$ , and sends  $a_n + S$  to Alice.
- Alice computes  $(*, a_n + S)(\varphi^m, a_m) = (*, \varphi^m(a_n + S)a_m)$ . She does not actually "compute" the first component of the pair. She only needs the second component of the pair, which is  $H^{-(m+n)}(HM)^{m+n} + (H^{-m}SH^m) \cdot (H^{-m}(HM)^m)$ . Since  $S \cdot (HM) = 0$ , so Alice gets  $K_{Alice} = a_{m+n}$ .
- Bob computes  $(*, a_m + R)(\varphi^n, a_n) = (*, \varphi^n(a_m + R)a_n)$ . He does not actually "compute" the first component of the pair. Similarly, he gets  $K_{Bob} = a_{m+n}$ .
- Alice and Bob have the shared secret key  $K = K_{Alice} = K_{Bob} = a_{m+n}$ .

It is shown in [4] that the linear algebra attack as above does not work against this protocol. Unfortunately, this protocol is vulnerable against the linear decomposition attack as follows.

Consider the linear space  $W$  generated by all elements of the form  $H^{-k}(HM)^k$  where  $k = 1, 2, \dots$ . Note that  $a_m, a_n \in W$ . Let  $U$  be the annihilator space of  $HM$  consisting of all matrices  $A \in \mathbf{A}$  such that  $A \cdot (HM) = 0$ . Note that  $R, S \in U$ . Let  $Z = W + U$ . One can find effectively a basis  $e_1, \dots, e_l, f_1, \dots, f_t$  of  $Z$ , where  $e_i \in W, i = 1, \dots, l$ ;  $f_j \in U, j = 1, \dots, t$ . Let  $e_i = H^{-k_i}(HM)^{k_i}$ , where  $k_i \in \mathbb{N}, i = 1, \dots, l$ .

Now we have public dates  $a_m + R$  and  $a_n + S$  where  $m, n \in \mathbb{N}$ , and we know that  $R, S \in U$ . We can effectively compute

$$(5) \quad a_n + S = \sum_{i=1}^l \eta_i e_i + \sum_{j=1}^t \nu_j f_j = \sum_{i=1}^l \eta_i (H^{-k_i}(HM)^{k_i}) + S_1,$$

where  $\eta_i, \nu_j \in \mathbb{F}$  for  $i = 1, \dots, l$  and  $j = 1, \dots, t$ , and  $S_1 \in U$ . It is possible that  $S_1 \neq S$ .

Then

$$\begin{aligned} \sum_{i=1}^l \eta_i H^{-k_i} (H^{-m} (HM)^m + R) (HA)^{k_i} = \\ = H^{-m} \left( \sum_{i=1}^l \eta_i H^{-k_i} (HA)^{k_i} \right) (HM)^m = \end{aligned}$$

$$(6) \quad H^{-m} (H^{-n} (HM)^n - S_1) (HM)^m = H^{-(m+n)} (HM)^{m+n} = a_{m+n}.$$

Thus one has the shared secret key  $K = a_{m+n}$ . Note: 1) the basis  $e_1, \dots, e_l, f_1, \dots, f_t$  is constructed one time offline, 2) we don't need to look in invertible solution of considered sets of linear equations along the algorithm works. We apply the usual Gauss elimination process to find unique solution every time when we solve sets of linear equations in the algorithm. Hence, this algorithm is deterministic. Moreover, in the case where the platform is such or similar as proposed in [4] the algorithm is practical. Note: we don't compute  $m$  and/or  $n$  to recover  $K$ .

#### REFERENCES

- [1] V.A. Roman'kov. Algebraic cryptography. Omsk, Omsk State Dostoevsky University, 2013, 135 p. (in Russian).
- [2] V.A. Roman'kov. Cryptanalysis of some schemes applying automorphisms. *Prikladnaya Discretnaya Matematika*. **3** (2013), 35-51 (in Russian).
- [3] M. Habeeb, D. Kahrobaei, C. Koupparis, V. Shpilrain. Public key exchange using semidirect product of (semi)groups. In: *ACNS 2013, Lecture Notes Comp. Sc.* **7954** (2013), 475-486.
- [4] D. Kahrobaei, H.T. Lam, V. Shpilrain. Public key exchange using extensions by endomorphisms and matrices over a Galois field. Preprint, 11 p.
- [5] V. Roman'kov, A. Myasnikov. A linear decomposition attack. arXiv 1412.6401v1 [math. GR] 19 Dec. 2014.
- [6] K.H. Ko, S.J. Lee, J.H. Cheon, J.W. Han, J. Kang, C. Park. New public-key cryptosystem using braid groups. In: *Advances in Cryptology - CRYPTO 2000* **1880** of Lecture Notes Comp. Sc., Berlin, 2000, Springer, 166-183.
- [7] L. Wang, L. Wang, Z. Cao, E. Okamoto, J. Shao. New constructions of public-key encryption schemes from conjugacy search problems. In: *Information security and cryptography*. **6584** of Lecture Notes Comp. Sc., Springer, 2010, 1-17.
- [8] E. Stickel. A New Method for Exchanging Secret Keys. In: *Proc. of the Third Intern. Conf. on Information Technology and Applications (ICITA 05). Contemp. Math.* **2** (2005), IEEE Computer Society, 426-430.
- [9] R. Alvarez, F.-M. Martinez, J. F. Vicent, A. Zamora. A Matricial Public Key Cryptosystem with Digital Signature. *WSEAS Trans. on Math.* **4**, No. 7 (2008), 195-204.
- [10] V. Shpilrain, A. Ushakov. A new key exchange protocol based on the decomposition problem. In: *Algebraic Methods in Cryptography*. **418** of Contemporary Mathematics, AMS, 2006, 161-167.
- [11] U. Romanczuk and V. Ustimenko. On the  $\text{PSL}_2(q)$ , Ramanujan graphs and key exchange protocols. Available at <http://aca2010.info/index.php/aca2010/aca2010/paper/viewFile/80/3>.
- [12] A. Mahalanobis. The Diffie-Hellman key exchange protocol and non-abelian nilpotent groups. *Israel J. Math.* **165** (2008), 161-187.

- [13] S.K. Rososhek. Cryptosystems in the automorphism groups of group rings of abelian groups. *Fundamentalnaya i prikladnaya matematika* **13** (2007), 157-164 (in Russian).
- [14] V.T. Markov, A.V. Mihalyov, A.V. Gribov, P.A. Zolotyh, S.S. Skazhenik. Quasigroups and rings in coding and cryptoschemes constructing. *Prikladnaya Discretnaya Matematika*. **4** (2012), 35-52 (in Russian).
- [15] V.A. Roman'kov. A polynomial time algorithm for the braid double shielded public key cryptosystems. arXiv 1412.5277v1 [math. GR] 17 Dec. 2014.
- [16] X. Wang, C. Xu, G. Li, H. Lin, W. Wang. Double shielded Public Key Cryptosystems. *Cryptology ePrint Archive Report 2014/588*, (2014), 1-14.

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